

OPE Constraints and the Leading Order Hadronic Contribution to $(g - 2)_\mu$

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OPE constraints are studied as a means of distinguishing between the versions of the $I = 1$ vector spectral function extracted from (i) inclusive $I = 1$ hadronic electroproduction cross-sections and (ii) inclusive $I = 1$ hadronic τ decay data, with the goal of clarifying expectations for the leading order hadronic contribution to $a_\mu = (g - 2)_\mu/2$ in the Standard Model. The constraints are shown to, at present, favor the τ decay data, and hence a Standard Model prediction for a_μ compatible with the BNL E821 experimental result. The relative role of the $\pi\pi$ and 4π contributions to the discrepancy between the integrated electroproduction results and the corresponding OPE constraints is also investigated, and the significance, in this context, of forthcoming re-measurements of the $e^+e^- \rightarrow \pi^+\pi^-\pi^0\pi^0$ cross-sections pointed out.

1. Background

It is well known that

- the leading order (LO) hadronic contribution to a_μ , $[a_\mu]^{had,LO}$, can be written as a dispersive integral

$$[a_\mu]^{had,LO} = \frac{\alpha_{EM}^2}{3\pi^2} \int_{4m_\pi^2}^{\infty} ds \frac{K(s)}{s} R(s) \quad (1)$$

with $K(s)$ a known function and

$$R(s) = \frac{3s \sigma[e^+e^- \rightarrow hadrons]}{16\pi \alpha_{EM}^2} \quad (2)$$

- as a consequence of CVC, the $I = 1$ parts of both $R(s)$ and the hadronic τ decay distribution provide experimental determinations of the $I = 1$ vector spectral function, $\rho^{I=1}(s)$, allowing τ decay data to, in principle, be incorporated into the determination of $[a_\mu]^{had,LO}$ [1]
- after known isospin-breaking (IB) corrections to the CVC relation have been performed, the current τ and electroproduction

(EM) versions of $\rho^{I=1}(s)$ do not agree [2], leading to incompatible determinations of $[a_\mu]^{had,LO}$ [3].

The τ -based determination corresponds to a SM value for a_μ compatible, within errors, with the BNL E821 experimental result [4], while the most recent assessment of the electroproduction (EM) determination leads to a value in $\sim 3\sigma$ disagreement with experiment [3].

The discrepancy between the measured EM version of $\rho^{I=1}(s)$ and that implied by IB-corrected τ decay data is now generated essentially entirely by contributions from the $\pi^+\pi^-$ and $\pi^+\pi^-\pi^0\pi^0$ states. The differences, moreover, are such that, where the two $\rho^{I=1}(s)$ determinations disagree, that based on the τ data lies higher. Figures of the preliminary BELLE $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ data [5] indicate that, while the region over which a discrepancy is observed to occur may be shifted by the BELLE results, it remains true that, where disagreement exists, it is the τ version which is higher.

As pointed out in Ref. [6], the uniformity in sign of $\rho_\tau^{I=1}(s) - \rho_{EM}^{I=1}(s)$ makes it rather easy to construct finite energy sum rules (FESR's) which distinguish between the two versions of $\rho^{I=1}(s)$.

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The form of these FESR's is

$$\int_0^{s_0} w(s) \rho(s) ds = -\frac{1}{2\pi} \oint_{|s|=s_0} w(s) \Pi(s) ds \quad (3)$$

where $w(s)$ is an analytic weight function, $\rho(s) \equiv \rho^{I=1}(s)$, and $\Pi(s)$ is the relevant (charged or neutral current) $I = 1$ vector current correlator. The parameter s_0 is to be chosen large enough that the OPE representation of Π can be used on the RHS of Eq. (3). For weights $w(y)$ which are (i) functions of the dimensionless variable $y = s/s_0$ and (ii) both non-negative and monotonically decreasing on the interval $0 \leq y \leq 1$, it follows that, if the τ data is correct, then both the slope with respect to s_0 and the magnitude of the EM spectral integrals *for all* s_0 will be too small, relative to OPE expectations. Similarly, if the EM data is correct, then both the magnitude and slope with respect to s_0 of the τ spectral integrals will be too large, relative to OPE expectations. It is worth stressing that, for the OPE integrals, the slope with respect to s_0 is considerably less sensitive than is the normalization to the main OPE input, α_s . We will return to this point in the discussion below.

In the analysis reported here we restrict our attention to $s_0 > 2 \text{ GeV}^2$, which choice strongly suppresses residual OPE breakdown for weights, $w(s)$, having a zero at $s = s_0$ [7,8]. At these scales, the OPE representation is strongly dominated by the $D = 0$ perturbative contribution, and hence well-determined once α_s is given at some particular reference scale. The needed input value is obtained by averaging the independent high-scale determinations of $\alpha_s(M_Z)$ reported in Ref. [9], which yields $\alpha_s(M_Z) = 0.1198 \pm 0.0020$. Values relevant to the lower scales required in this analysis are then obtained via 4-loop running and matching [11]. Details of the relevant input for, and treatment of, higher dimension OPE contributions may be found in Ref. [6]. The weights used below, $\hat{w}(y) = 1 - y$ and $w_N(y) = 1 - \left(\frac{N}{N-1}\right)y + \left(\frac{1}{N-1}\right)y^N$, $N = 3, \dots, 6$, are chosen to strongly reduce sensitivity to potential poorly known $D \geq 6$ OPE contributions [6]. The $w_N(y)$ also have a double zero at $s = s_0$, further suppressing possible residual OPE breaking contributions.

The $I = 1$ hadronic τ decay distribution has been measured by ALEPH [12], CLEO [13] and OPAL [14]. The results below are based on the ALEPH determination, for which the covariance matrix has been made publicly accessible. Additional corrections for long-distance EM effects, as evaluated in Refs. [15], are applied to the $\pi\pi$ component of the τ data [16]. For pre-2003 EM data, we employ the results for various exclusive modes reported in the compilation of Ref. [17]. Small missing modes are accounted for using isospin relations and the techniques described in detail for such modes in Ref. [2]. Post-2003 updates for a number of modes [18] have also been taken into account. Where uncertainties exist in older publications concerning the treatment of radiative corrections and/or systematic errors, and newer data with no such uncertainties are available, we employ only the newer data.

2. Results

Examples of the results of the comparison between the spectral integrals and OPE expectations, for the EM case, are shown in Figures 1 and 2, for the weights $\hat{w}(y)$ and $w_6(y)$, respectively. Analogous results for the IB-corrected τ case are shown in Figures 3 and 4, respectively. We see that the normalization and slope of the τ spectral integrals are in excellent agreement with OPE expectations, while both the normalization and slope are low in the EM case. The same behavior is found for FESR's based on other non-negative, monotonically decreasing weights, though for brevity we have displayed only the $\hat{w}(y)$ and $w_6(y)$ results. As noted above, this pattern is the one expected if it is the τ data which are correct and the EM data which are wrong. (Note, however, that an alternate τ spectral distribution which, like the preliminary BELLE $\tau \pi\pi$ distribution, is larger for some s but smaller for other s than is the corresponding ALEPH distribution, is also capable of satisfying the OPE constraints; i.e., the fact that the constraints are satisfied by the ALEPH data does not necessarily imply that the ALEPH data set is correct.)

We now quantify the extent of the disagreement between the EM spectral integrals and corre-

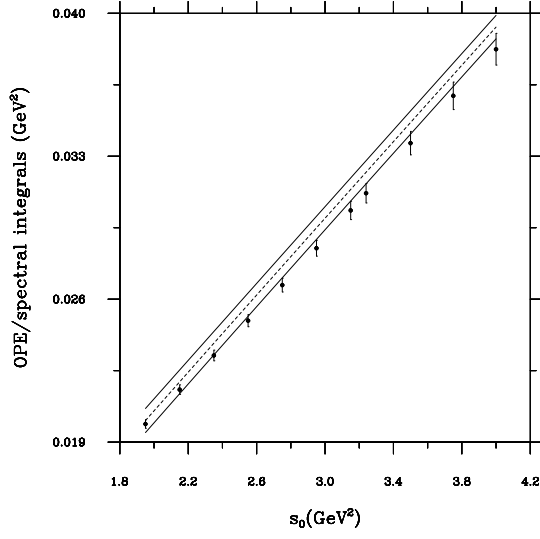
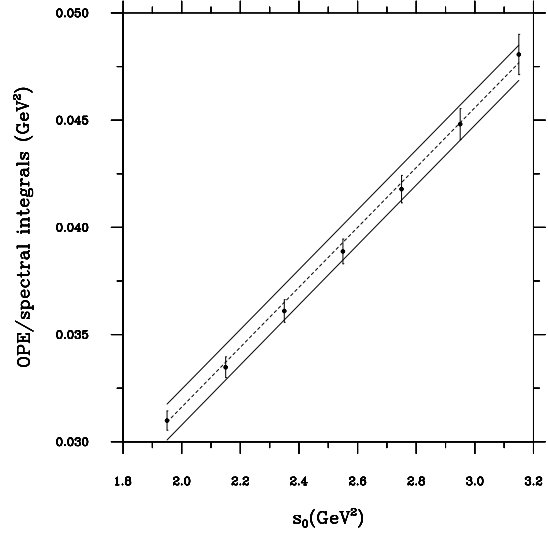
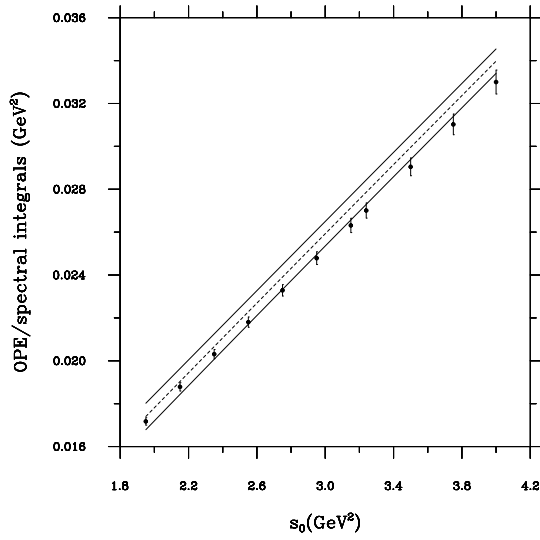
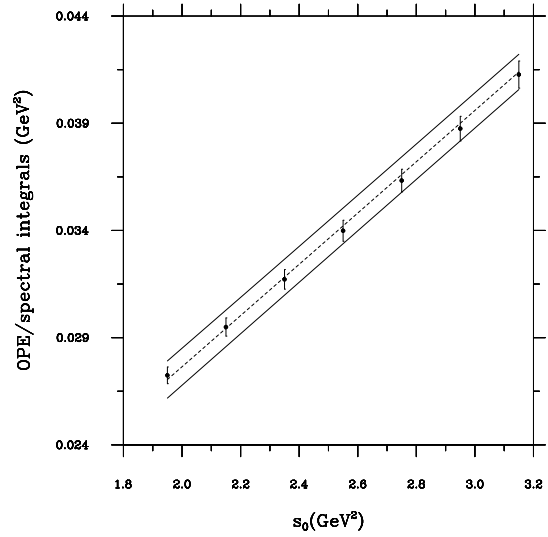
Figure 1. EM OPE and spectral integrals for $\hat{w}(y)$ Figure 3. τ OPE and spectral integrals for $\hat{w}(y)$ Figure 2. EM OPE and spectral integrals for $w_6(y)$ Figure 4. τ OPE and spectral integrals for $w_6(y)$ 

Table 1
 $\alpha_s(M_Z)$ from a fit to the EM and τ spectral integrals at $s_0 \sim 4 \text{ GeV}^2$ and $s_0 \sim m_\tau^2$, respectively

$w(y)$	EM or τ	$\alpha_s(M_Z)$
\hat{w}	EM	$0.1138^{+0.0030}_{-0.0035}$
w_3	EM	$0.1152^{+0.0019}_{-0.0021}$
w_6	EM	$0.1150^{+0.0022}_{-0.0026}$
$1 - y$	τ	$0.1212^{+0.0027}_{-0.0032}$
w_3	τ	$0.1189^{+0.0018}_{-0.0021}$
w_6	τ	$0.1195^{+0.0020}_{-0.0022}$

Table 2
The slopes with respect to s_0 of the EM OPE and spectral integrals. The first and second line OPE entries for each weight correspond to the independent high-scale average and fitted values of $\alpha_s(M_Z)$, respectively, as described in the text.

$w(y)$	S_{exp}	S_{OPE}
\hat{w}	$.00872 \pm .00026$	$.00943 \pm .00008$ $.00934 \pm .00008$
w_6	$.00762 \pm .00017$	$.00811 \pm .00009$ $.00805 \pm .00009$

Table 3
Shifts in the effective EM spectral integrals associated with the $\pi\pi$ and 4π parts of the EM $\rightarrow \tau$ spectral data replacement, for $s_0 = 2 \text{ GeV}^2 [m_\tau^2]$

$w(y)$	$\pi\pi$	4π
\hat{w}	82% [36%]	18% [64%]
w_6	87% [45%]	13% [55%]

sponding OPE expectations. As a measure of the normalization disagreement, we fit $\alpha_s(M_Z)$ to the measured EM spectral integrals at $s_0 \sim 4 \text{ GeV}^2$ (the highest scale for which it is still possible to construct the EM spectral function as a sum over observed exclusive modes) and compare this result to the independent high-scale average noted above. A similar fit is performed for the τ data, in this case at the maximum scale, $s_0 = m_\tau^2$, accessible in hadronic τ decay. The results of these exercises are shown in Table 1. The fitted values are seen to be in good agreement with the high-scale average in the τ case, but $\sim 2 - 2.5 \sigma$ low in the EM case. Results in agreement with those shown in the table are also obtained for other non-negative, monotonically decreasing weights [19].

The results for the EM spectral integral and OPE slopes with respect to s_0 (S_{exp} and S_{OPE} , respectively) are shown in Table 2. In the OPE case two values are given for each weight, that in the first row corresponding to the independent high-scale average $\alpha_s(M_Z) = 0.1198 \pm 0.0020$, that in the second row to the fitted value obtained above for the weight in question, and given already in Table 1. We see explicitly, as noted above, that the OPE slope is *very* insensitive to $\alpha_s(M_Z)$. As is evident from the table, no realistic value for $\alpha_s(M_Z)$ will suffice to bring the OPE and spectral integral slopes into agreement; such agreement can only be obtained through changes in the experimental spectral function. The slope discrepancies are at the $\sim 2.5 \sigma$ level.

To investigate the extent to which the source of the normalization and slope disagreement in the EM case lies in the $I = 1$, as opposed to $I = 0$, portion of the spectral function, we replace the EM $\pi\pi$ and 4π data with the corresponding IB-corrected τ results, and rerun the OPE/spectral integral comparison. It is found that both the slope and normalization of the resulting τ -modified “EM” spectral integrals are in excellent agreement with the OPE constraints. An illustration of this point, for the w_6 FESR, is given in Figure 5.

Finally, we consider the relative role of the $\pi\pi$ and 4π components of the $I = 1$ τ -EM spectral function difference in producing the agreement between the OPE constraints and the τ -modified

Figure 5. OPE and τ -modified “EM” spectral integrals for $w_6(y)$. The solid circles (with error bars) represent the original EM spectral integrals, the open circles the τ -modified “EM” ones.

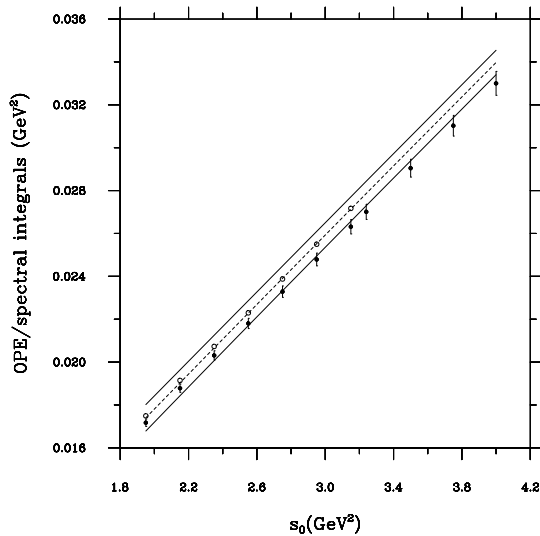


Table 4

Impact on the EM slope with respect to s_0 of the replacement of EM 4π data with the equivalent τ data, for \hat{w} and w_6 . The modified EM experimental slope (*exp*) is given in the first line, the OPE slope (*OPE*) in the second.

$w(y)$	$\alpha_s(M_Z)$	Slope
\hat{w}	.1186	.00936 \pm .00026 (exp)
		.00940 \pm .00008 (OPE)
w_6	.1176	.00795 \pm .00017 (exp)
		.00808 \pm .00009 (OPE)

“EM” spectral integrals. Since $[a_\mu]^{had,LO}$ is more strongly dominated by the $\pi\pi$ component of the spectral function than are the spectral integrals appearing on the LHS’s of the FESR’s employed in our analysis, this question is of relevance to determining the implications of our results for $[a_\mu]^{had,LO}$.

The relative $\pi\pi$ and 4π contributions to the spectral integral shifts caused by the $\rho_{EM}(s) \rightarrow \rho_\tau(s)$ replacement are shown, for our two representative weights, in Table 3. The results for the other w_N are similar to those for w_6 . The $\pi\pi$ component is seen to dominate for $s_0 \sim 2 \text{ GeV}^2$ and remain important for $s_0 \sim m_\tau^2$.

Since the agreement between different experiments for the EM $\pi^+\pi^-\pi^0\pi^0$ cross-sections is, at present, not good [2], it is of relevance to consider the impact of replacing only the 4π component of the EM spectral function with the corresponding τ component. The impact of this change on the normalization and slope of the EM spectral integrals is shown in Table 4. We see that, were future results to bring the EM 4π cross-sections into agreement with expectation based on τ decay data, the difference between the slope and normalization of the EM spectral integrals and the OPE constraint values would be reduced to the $\sim 1\sigma$ level. Such a change would also reduce the discrepancy between the EM-data-based SM prediction for a_μ and both the τ -data-based SM prediction and the E821 experimental result [20]. The discrepancy between the τ and EM versions of $[a_\mu]^{had,LO}$ would also be reduced were the $\sim 4 \times 10^{-10}$ reduction, relative to the earlier τ average, seen in the preliminary BELLE result for the $\pi\pi$ contribution, to remain present in the final version of the analysis. Such an improved EM- τ consistency, according to the results of this study, would almost certainly be accompanied by a reduction in the difference between the experimental a_μ value and the EM-based SM prediction. Such a development would make even more important the role of a reduced experimental uncertainty [22] in determining whether or not beyond-the-SM contributions have been detected in a_μ .

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19. The results obtained for $\alpha_s(M_Z)$ using the w_3, w_4 , and w_5 sum rules are, for the EM case, $0.1152^{+0.0019}_{-0.0021}$, $0.1154^{+0.0020}_{-0.0023}$, and $0.1152^{+0.0022}_{-0.0024}$, respectively, and, for the τ case, $0.1189^{+0.0018}_{-0.0021}$, $0.1193^{+0.0019}_{-0.0022}$, and $0.1194^{+0.0020}_{-0.0022}$, respectively. All are in excellent agreement with the corresponding result from the w_6 sum rule.
20. A further small reduction in the difference between the EM and τ versions of $[a_\mu]^{had, LO}$, not incorporated into the recent update of Refs. [3], is associated with the model dependence of the " ρ - ω mixing" contribution to the IB correction. In Ref. [21], it was shown that the model dependence encountered in attempting to separate the IB interference component of the EM $\pi\pi$ cross-sections is significantly larger than the uncertainty produced by fitting any given model to the data. Those models incorporating more in the way of known QCD constraints (the GS and GP/CEN* models in the notation of Ref. [21]) yield values of the integrated EM ρ - ω interference contribution smaller than that of Refs. [15] (the version employed in Refs. [3]) by $\sim 10\%$ of the discrepancy between the "standard" assessment of the τ and EM $\pi\pi$ contributions to $[a_\mu]^{had, LO}$. This model dependence also produces an increase in the combined uncertainty of the overall IB correction. This increased uncertainty is also not taken into account in various recent assessments of the EM- τ comparison.
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